1 Introduction to limits

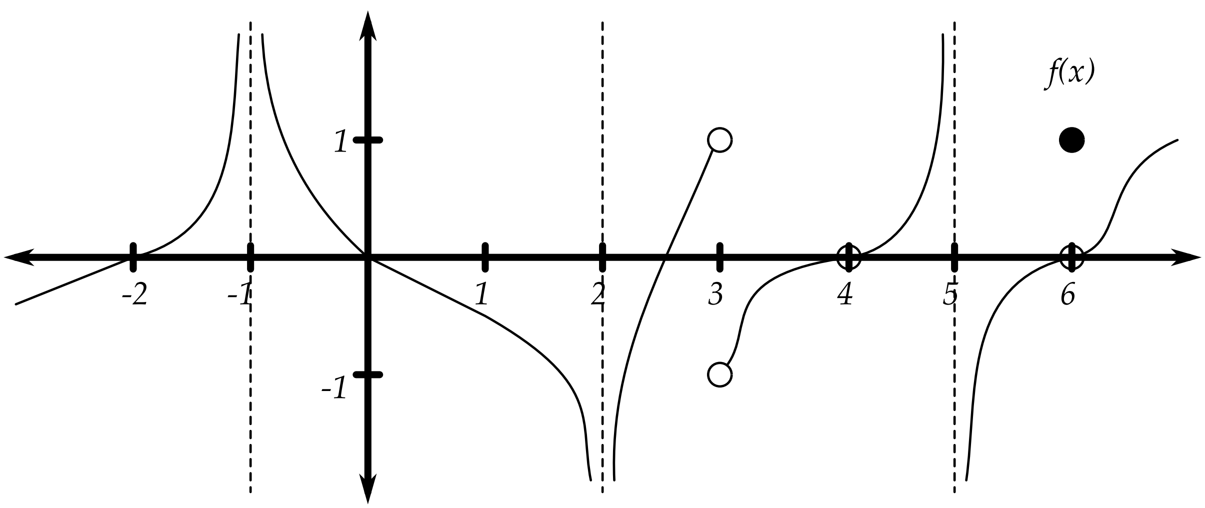
1.1 Introduction to limits

1. Numerically compute the following limits.

   (a) $\lim_{x \to 4} 2x - 3$

   (b) $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 + x - 12}$

   (c) $\lim_{x \to 0} \dfrac{\sin(3x)}{4x}$



2. Find the following limits, if they exist.

$\lim_{x \to -1} f(x) =$ $\quad$ $\lim_{x \to 0} f(x) =$ $\quad$ $\lim_{x \to 2} f(x) =$ $\quad$ $\lim_{x \to 3} f(x) =$

$\lim_{x \to -1^+} f(x) =$ $\quad$ $\lim_{x \to 0^+} f(x) =$ $\quad$ $\lim_{x \to 2^+} f(x) =$ $\quad$ $\lim_{x \to 3^+} f(x) =$

$\lim_{x \to -1^-} f(x) =$ $\quad$ $\lim_{x \to 0^-} f(x) =$ $\quad$ $\lim_{x \to 2^-} f(x) =$ $\quad$ $\lim_{x \to 3^-} f(x) =$

$\lim_{x \to 4} f(x) =$ $\quad$ $\lim_{x \to 5} f(x) =$ $\quad$ $\lim_{x \to 6} f(x) =$

$\lim_{x \to 4^+} f(x) =$ $\quad$ $\lim_{x \to 5^+} f(x) =$ $\quad$ $\lim_{x \to 6^+} f(x) =$

$\lim_{x \to 4^-} f(x) =$ $\quad$ $\lim_{x \to 5^-} f(x) =$ $\quad$ $\lim_{x \to 6^-} f(x) =$

1.2 Basic limits and limit laws

1. Use the limit laws to find the following limits.

   (a) $\lim_{x \to 3} x^2 + 3x + 7$

(b) $\lim_{x \to 4} \dfrac{\sqrt{x} + 7x^2}{x^2 + 4\sqrt{x}}$

(c) $\lim_{x \to 2} x^2(x^3 - \sqrt{x})$

(d) $\lim_{x \to 3} \dfrac{x\sqrt{2x^2 + 7}}{x - 1}$

2. If $\lim_{t \to 2} f(t) = -2$ and $\lim_{t \to 2} g(t) = 5$, use the limit laws to find:

(a) $\lim_{t \to 2} \dfrac{2g(t) - 4}{f(t) + g(t)}$

(b) $\lim_{t \to 2} \dfrac{f(t) - 3g(t)}{2g(t) + 7}$

1.3 Using limit laws

1. Find the following limits.

(a) $\lim_{x \to 3} 2x^4 - 7x^2 + 5$

(b) $\lim_{x \to -5} \dfrac{x^2 - 5}{7 - x}$

2 Evaluating limits analytically

2.1 Limits of the form 0/0

1. Find the following limits.

(a) $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

(b) $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 + x - 12}$

(c) $\lim_{x \to -1} \dfrac{x^3 + 3x^2 + 3x + 1}{x^2 - 1}$

(d) $\lim_{x \to 2} \dfrac{2x^2 - 3x - 2}{x^2 + 3x - 10}$

2. Find the following limits.

(a) $\lim_{x \to 2} \dfrac{3x - 6}{1 - \sqrt{x - 1}}$

(b) $\lim_{h \to 0} \dfrac{\sqrt{9 + h} - 3}{h}$

(c) $\lim_{t \to 0} \dfrac{t}{\sqrt{1 + 3t} - 1}$

(d) $\lim_{x\to 3} \dfrac{\sqrt{7-x}-2}{3x-9}$

3. Find the following limits.

   (a) $\lim_{x\to 0} \dfrac{(3+x)^{-1}-3^{-1}}{x}$

   (b) $\lim_{x\to 1} \dfrac{1}{x-1} - \dfrac{2}{x^2-1}$

   (c) $\lim_{x\to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$

   (d) $\lim_{x\to 0} \dfrac{2(1+x)^2-2}{x}$

   (e) $\lim_{x\to 0} \dfrac{e^{3x}-e^x}{1-e^x}$

   (f) $\lim_{\theta\to 0} \dfrac{\cos(\theta)-1}{\sin(\theta)}$

2.2 The Squeeze Theorem and special limits

1. Find the following limits.

   (a) $\lim_{x\to 0} \dfrac{\sin(x)}{2x}$

   (b) $\lim_{x\to 0} \dfrac{x}{\sin(x)}$

   (c) $\lim_{x\to 0} \dfrac{\sin(3x)}{3x}$

   (d) $\lim_{x\to 0} \dfrac{5x}{\sin(5x)}$

   (e) $\lim_{x\to 0} \dfrac{\sin(3x)}{4x}$

   (f) $\lim_{x\to 0} \dfrac{\tan(x)}{x}$

   (g) $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(3x)}$

   (h) $\lim_{x\to 0} \dfrac{\sqrt{x+2}\,\sin(x)}{x}$

   (i) $\lim_{x\to 0} \dfrac{\tan(3x)}{\sin(4x)}$

   (j) $\lim_{x\to 0} \dfrac{\sin^2(4x)}{3x^2\cos^2(5x)}$

   (k) $\lim_{x\to 0} \dfrac{\sin(x)}{1-\sqrt{1+x}}$

2. Use the Squeeze Theorem to prove that

   (a) $\lim_{x\to 0} x^2 \sin\left(\frac{78}{x^4}\right) = 0$

   (b) $\lim_{x\to 3} (x-3)^2 \cos\left(\frac{\pi}{x-3}\right) = 0$

2.3 One-sided limits

1. Find the following limits, using one-sided limits as appropriate.

   (a) $\lim_{x\to 3^-} 2x+1$

   (b) $\lim_{x\to 3^+} \frac{x^2-9}{x-3}$

   (c) $\lim_{x\to 3} \frac{|x-3|}{x-3}$

   (d) $\lim_{x\to -4} \frac{|x^2-16|}{|x+4|}$

   (e) $\lim_{x\to 7} \frac{|7-x|}{x^2-49}$

   (f) $\lim_{x\to 7} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 7, \\ 3x+1 & \text{if } x \geq 7. \end{cases}$

   (g) $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 7, \\ 3x+1 & \text{if } x \geq 7. \end{cases}$

   (h) $\lim_{x\to 7} g(x)$, where $g(x) = \begin{cases} 2x & \text{if } x < 7, \\ 3x-7 & \text{if } x \geq 7. \end{cases}$

   (i) $\lim_{x\to 0} h(x)$, where $h(x) = \begin{cases} x & \text{if } 0 \leq x < 5, \\ \sin(x) & \text{if } x < 0, \\ 2x-7 & \text{if } x > 5. \end{cases}$

   (j) $\lim_{x\to 5} h(x)$, where $h(x) = \begin{cases} x & \text{if } 0 \leq x < 5, \\ \sin(x) & \text{if } x < 0, \\ 2x-7 & \text{if } x > 5. \end{cases}$

   (k) $\lim_{x\to 3} x - \lfloor x \rfloor$

3 Continuity and infinite limits

3.1 Continuity at a point

1. Determine if the given function is continuous at the given point.

   (a) $f(x) = x^2 + 3x + 4$, at the point $x = 3$

   (b) $g(x) = \frac{3}{x-4}$, at the point $x = 4$

(c) $h(x) = \dfrac{x+1}{x+1}$, at the point $x = -1$

(d) $k(x) = \begin{cases} 3 & \text{if } x = 0 \\ x+1 & \text{if } x \neq 0 \end{cases}$, at the point $x = 0$

(e) $m(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$, at the point $x = 0$

3.2 Continuous functions

1. Prove that:

   (a) $f(x) = \sin(x) + \dfrac{x^2}{x^2+1}$ is continuous everywhere.

   (b) $g(x) = \cos(x^2 + 1)$ is continuous everywhere.

   (c) $h(x) = x^2 \sin^3(x^4)$ is continuous everywhere.

2. (a) Use the Intermediate Value Theorem to show that $f(x) = x^3 - 5x + 1$ has a root between 2 and 3.

   (b) Use the Intermediate Value Theorem to show that $g(x) = x^3 + 3x^2 - 4x - 11$ has three roots in $[-4, 4]$.

   (c) Use the Intermediate Value Theorem to show that $h(x) = x^3 - x^2 - 4x + 2$ has three roots in $[-3, 3]$.

3.3 Types of discontinuities

1. Find the following limits, if they exist. If not, indicate $\pm\infty$ where applicable.

   (a) $\displaystyle\lim_{x \to -1^+} \frac{1}{x+1}$

   (b) $\displaystyle\lim_{x \to -2} \frac{x}{2+x}$

   (c) $\displaystyle\lim_{x \to -2} \frac{x}{(2+x)^2}$

   (d) $\displaystyle\lim_{x \to -2} \frac{-x}{(2+x)^2}$

   (e) $\displaystyle\lim_{x \to 1} \frac{x^2+5x+6}{x^2-3x+2}$

   (f) $\displaystyle\lim_{x \to 4} \frac{x^2+3x+2}{x^2-8x+16}$

   (g) $\displaystyle\lim_{x \to 4} \frac{x^2+3x+2}{x^2-x-12}$

2. Classify the following discontinuities.

   (a) $f(x) = \lfloor x \rfloor$ at $x = 3$

(b) $g(x) = \dfrac{x^2 - 1}{x - 1}$ at $x = 1$

(c) $h(x) = \dfrac{1}{3 - x}$ at $x = 3$

3. For each of the following functions, determine the points at which the function is discontinuous and state the type of discontinuity: removable or irremovable, and if the latter, jump or infinite.

   (a) $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 9}$

   (b) $f(x) = \sec(x)$

   (c) $f(x) = \dfrac{\sin(x)}{x}$

   (d) $f(x) = \begin{cases} \frac{x+2}{x^2-4} & \text{if } x \neq -2 \\ -\frac{1}{4} & \text{if } x = -2 \end{cases}$

   (e) $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

   (f) $f(x) = x - \lfloor x \rfloor$

4. Find all values c that make this function continuous everywhere.

$$f(x) = \begin{cases} cx^2 - 3x + 7 & \text{if } x \leq 2 \\ 2x^2 + cx + 5 & \text{if } x > 2 \end{cases}$$

5. Find the vertical asymptotes of the following functions.

   (a) $f(x) = \dfrac{x^2 + 4x + 4}{x^2 + 4x + 3}$

   (b) $g(x) = \dfrac{x^2 + 6x + 9}{x^2 - 9}$

   (c) $h(x) = 7x + \dfrac{1}{x - 3}$

4 The derivative and the tangent line problem

4.1 Definition of derivative at a point

1. Find the following, if they exist.

   (a) $f'(3)$ if $f(x) = x^2 + 2x + 1$.

   (b) $g'(7)$, if $g(x) = \sqrt{x + 2}$.

   (c) $k'(0)$, if $k(x) = |x|$.

2. Find the equation of the tangent line through the given point for each of the following.

   (a) $f(x) = x^2 + 3x + 9$, at $P(-2, 7)$

   (b) $g(x) = 2x^2 - 6x + 5$, at $Q(1, 1)$

4.2 Definition of derivative as a function

1. Find the derivatives of the following functions.

   (a) $f(x) = x^2 + 3x$ (b) $f(x) = \dfrac{x+2}{x-3}$ (c) $f(x) = \sqrt{x+3}$

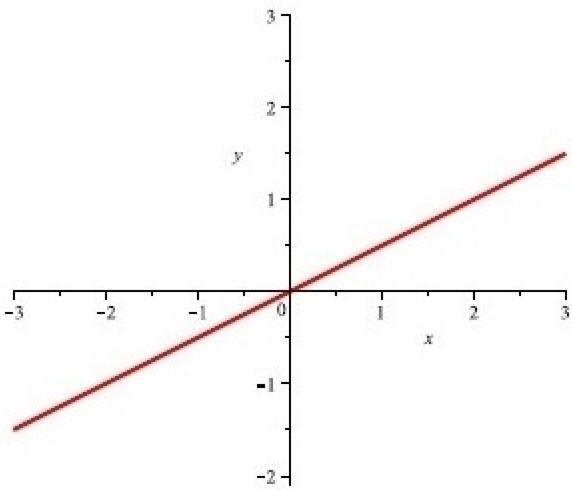
2. Find the slope of the tangent line to $f(x) = 3x^2 + 2x$ at

   (a) $x = 2$ (b) $x = -5$

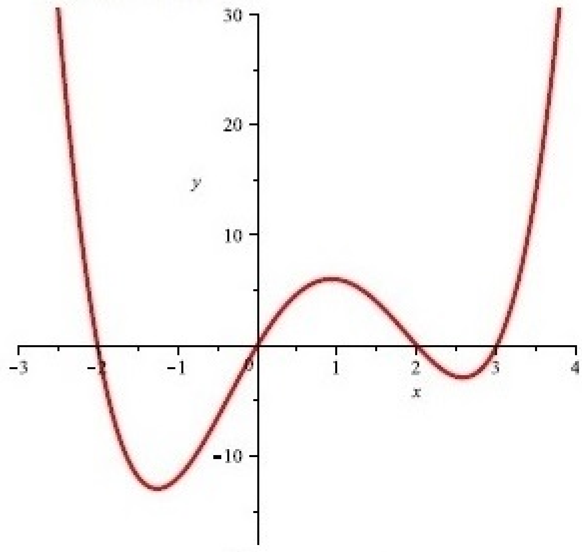
Graphs of *f(x)* and *f '(x)*

1. Draw a function which would have a derivative the same shape as the one pictured.

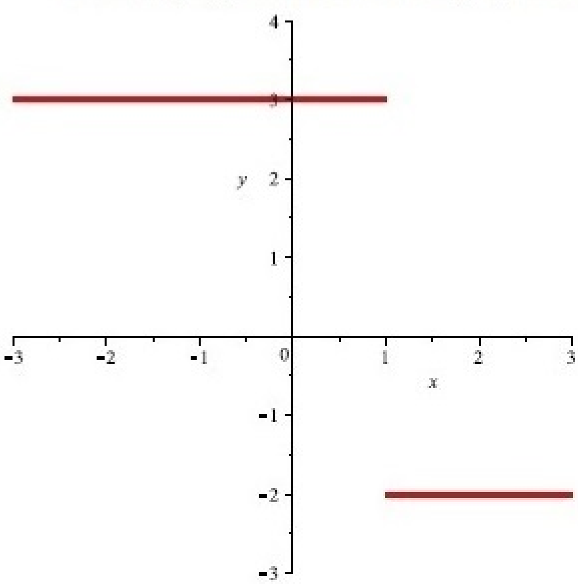
a) Sketch a function *f(x)* which would have a derivative *f '(x)* that looks like the graph below.



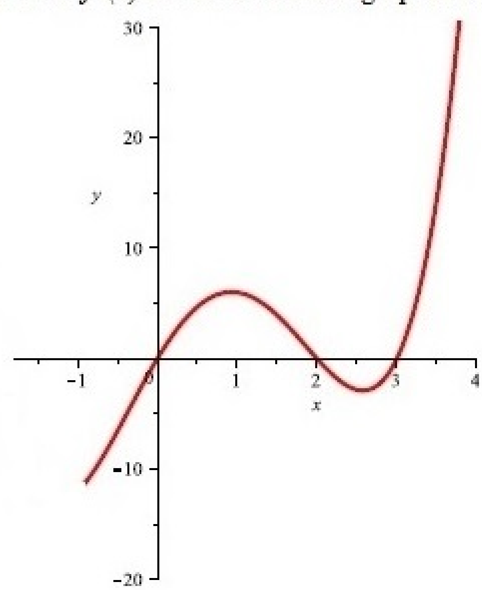
b) Sketch the derivative *f '(x)* for the function *f(x)* shown below.



c) Sketch a function *f(x)* which would have a derivative *f '(x)* that looks like the graph below.

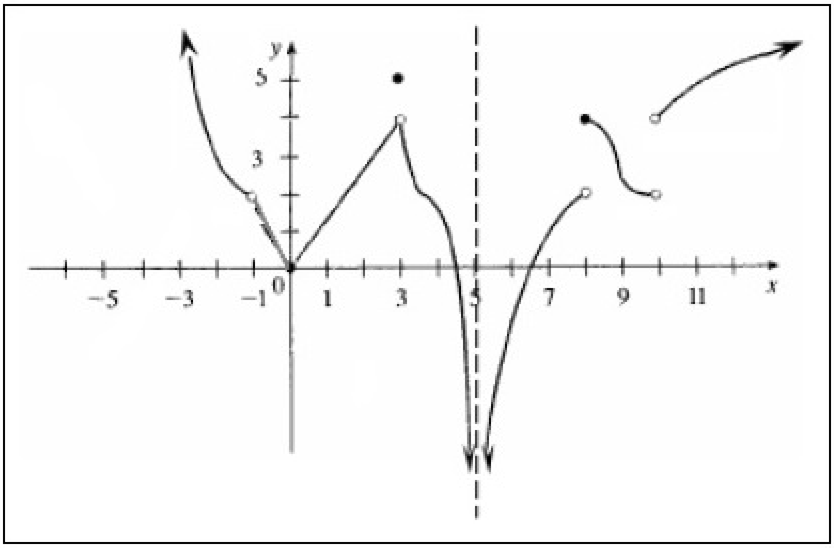


d) Sketch a function *f(x)* which would have a derivative *f '(x)* that looks like the graph below.



Differentiability

1. For function *f (x)* graphed below, please complete the following table.



	Continuous ?	Differentiable ?	
	Yes / No ?	Yes / No ?	Why ?
a = -1			
a = 0			
a = 3			
a = 4			
a = 5			
a = 9			

2. Is the function $f(x) = |x^2 - 4x + 3|$...

   (a) ...continuous at $x = 3$?

   (b) ...differentiable at $x = 3$?

   (c) ...continuous at $x = 2$?

   (d) ...differentiable at $x = 2$?

4.3 Basic Rules

1. Find the derivative.

   (a) $f(x) = x^7$

   (b) $y = x^{312}$

   (c) $f(x) = x^{-5}$

   (d) $f(x) = \dfrac{1}{x^7}$

   (e) $f(t) = \sqrt{t}$

   (f) $y = \sqrt{x^5}$

   (g) $y = \sqrt[3]{x}$

   (h) $f(t) = \dfrac{1}{\sqrt[3]{t^2}}$

   (i) $f(x) = 5x^2 - 3x + 4$

   (j) $f(\theta) = \theta^2 + 4\theta - 7$

   (k) $f(w) = 3w^4 + 3w^3 - w + w^{-3}$

   (l) $f(x) = \dfrac{2}{x^3} - \dfrac{5}{x^2} + 3x^2\sqrt{x}$

   (m) $y = x^2(x^2+1)^2$

   (n) $f(t) = (t+3)^2(t-3)^2$

   (o) $y = \dfrac{2x^2 - x + \dfrac{7}{x^2}}{x}$

   (p) $f(x) = \dfrac{7 - x^2}{x} + \dfrac{\dfrac{1}{x} + x^2}{5}$

5 Rules of differentiation

5.1 Derivatives of trigonometric and exponential functions

1. Find the derivative.

   (a) $f(x) = 7\sin(x) + 3$

(b) $y = \frac{5}{\csc(x)}$

(c) $h(t) = 7t + t^2 + 5 + \cos(t)$

(d) $y = \dfrac{t^2+7}{t} + 6\sin(t)$

(e) $f(x) = \dfrac{x\cos(x)}{x} - 5x^2 + \dfrac{7}{x^5}$

2. Find the derivative.

(a) $f(r) = 7e^r$

(b) $y = 5x + 7x^2 - 3e^x$

(c) $g(t) = e^3 - 5e^t + t^2 - 7t + t^e$

(d) $y = 3\sin(x) + -5x^3 - 4e^x$

(e) $f(x) = \dfrac{7}{e^{-x}} + 5\cos(x) + \pi^2 + e^x$

5.2 The Product Rule

1. Find the derivatives.

(a) $f(x) = (3x+2)(2x+1)$

(b) $f(x) = xe^x$

(c) $f(x) = \sqrt{x}(x^3 + 4x + 3)$

(d) $f(x) = x^2\sin(x)$

(e) $f(x) = x^2e^x\sin(x)$

(f) $f(x) = x^2\sin^2(x)$

5.3 The quotient rule

1. Find the derivatives.

(a) $f(x) = \dfrac{x}{e^x}$

(b) $f(x) = \dfrac{x^2+3}{7-x^2}$

(c) $f(x) = \dfrac{\cos(x)}{3x^2+2}$

(d) $f(x) = \dfrac{e^x}{1+e^x}$

(e) $f(x) = \dfrac{x^2+1}{x\sin(x)}$

(f) $f(x) = \dfrac{x\cos(x)}{e^x\sin(x)}$

6 More rules of differentiation

6.1 Other trigonometric functions and higher derivatives

1. Find the derivatives.

   (a) $f(x) = 7x + \tan(x)$

   (b) $y = \sin^2(x)$

   (c) $h(t) = \dfrac{\csc(t) - 3}{t}$

   (d) $y = \dfrac{\sec(x)\tan(x)}{\cos(x)}$

   (e) $f(x) = xe^x \cos(x)$

   (f) $f(z) = \dfrac{z^2 \sin(z)}{\cos(z) - 5z}$

2. For each of the following, find the indicated derivative.

   (a) $f(x) = 3x^4 + 2x^2 + x + \ln(5x) + 3x^{-3}$, $f^{(5)}(x)$

   (b) $f(x) = \sin(-3x)$, $f^{(5)}(x)$

   (c) $y = \dfrac{\cos(x^2)}{e^x}$, y''

6.2 The chain rule

1. Find the derivative.

   (a) $y = (4 - 2x - 3x^2)^{78}$

   (b) $y = \tan\left(\theta^3 - 4\theta + 7\right)$

   (c) $f(x) = e^{e^x}$

   (d) $y = \dfrac{1}{\tan(x) - \cot(x)}$

   (e) $f(x) = \sin^3\left(e^{3x}\right)$

   (f) $g(t) = \left(\sqrt{t+2} - 2\right)^{\frac{3}{2}}$

   (g) $y = \sin(\cos(\sin(x)))$

   (h) $f(x) = \sin\left(xe^{-3x}\right)$

   (i) $y = \tan\left(\dfrac{x+1}{x^2+x}\right)$

   (j) $f(x) = \dfrac{7}{(1-3x)\sqrt{2-x}}$

   (k) $y = e^{x\sin(3x)}$

7 Implicit and logarithmic differentiation

7.1 Implicit differentiation

1. Find $\frac{dy}{dx}$.

   (a) $y^3 - 2y = 5x - 3x^2$

   (b) $\sqrt{x} + \sqrt[3]{y} = 5y$

   (c) $\cos(xy) = y$

   (d) $\sqrt{x+y} = \dfrac{1}{x} + \dfrac{1}{y}$

   (e) $\cot(xy^2) = (x+y)^2$

2. For each of the following, find the equation of the tangent line at the given point.

   (a) $\sin(x-y) = x\cos\left(y + \dfrac{\pi}{4}\right)$ at $\left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

   (b) $7xy + \cos(y) = 2x$ at $\left(0, \dfrac{\pi}{2}\right)$

3. (a) If $xy = x + y$, show that $y'' = \dfrac{2(y-1)}{(x-1)^2}$.

   (b) If $3xy + y^2 = 5x$, show that $\dfrac{d^2y}{dx^2} = -\dfrac{50}{(3x^2y)^3}$.

4. Find the derivative.

   (a) $y = \ln(x) + x^2$

   (b) $y = 7x^3 + 5x^2 + x + \ln(7)$

   (c) $f(x) = x^2 + e^x + \log_3(x)$

   (d) $y = \ln(x^3) + 7\cos(x) - \dfrac{7}{x}$

   (e) $g(t) = \ln(7x) + \log_3(5) + x^3$

   (f) $f(x) = 7^x$

   (g) $f(x) = x^2 7^x$

   (h) $f(x) = (7^x)^2$

   (i) $f(x) = 7^{x^2}$

   (j) $f(x) = 3e^x - x^e + 7^x$

   (k) $y = e^{\ln(x^2)}$

   (l) $f(z) = \ln(e^z - 8z)$

   (m) $f(x) = \ln\left(e^{\sec(x)}\right)$

   (n) $y = \ln\left((7-x)^3(2x+3)^2\right)$

(o) $y = \ln\left(\dfrac{\sqrt{x}}{e^x \tan^5(x)}\right)$

(p) $y = \ln((\ln(x))^3)$

(q) $y = \log_5(x^3 + 3)$

7.2 Logarithmic differentiation

1. Use logarithmic differentiation to find the derivative.

   (a) $y = (x^2 + 1)(x^4 + 5)(x^3 + 2)^2$

   (b) $y = \dfrac{x(x+1)^3}{(3x-1)^2}$

   (c) $y = \dfrac{5^x \sin^4(x)}{\sqrt{3x^5 - 7x}}$

   (d) $y = x^{\cos(x)}$

   (e) $y = x^{x^2}$

   (f) $y = (\ln(x))^x$

   (g) $y = (\sin(x))^{\sin(x)}$

   (h) $y = x^{x^x}$

7.3 Inverse trigonometric functions and their derivatives

1. Find the derivatives.

   (a) $y = 2\arccos(x)$

   (b) $f(x) = 3\sec(x) + 3\sec^{-1}(x)$

   (c) $f(t) = -\dfrac{3 + 5\arcsin(x)}{4}$

   (d) $y = \arctan(x)$

   (e) $y = \cot^{-1}\left(\sqrt{x}\right)$

   (f) $f(x) = \arccos\left(e^{2x}\right)$

2. Find the derivatives.

   (a) $f(x) = -\dfrac{\operatorname{sech}(x) + \sinh(x)}{5}$

   (b) $y = \tanh(1 - 4t)$

   (c) $y = \cosh(\ln(x))$

   (d) $f(x) = \arctan(\tanh(x^2))$

   (e) $y = \operatorname{sech}(\tan(\sqrt{x^3 + 1}))$

8 Related rates

8.1 Rectilinear motion

1. A particle's position is defined by $s(t) = 2t^3 - 6t^2 + 7t + 4$, where position is measured in meters and time is measured in seconds.

   (a) Find an expression for the particle's velocity, $v(t)$, and acceleration, $a(t)$.

   (b) What is the particle's average velocity between $t = 0$ and $t = 2$?

   (c) When is the particle's velocity 1 m/s?

   (d) When is the particle's acceleration 12 m/s^2?

2. A particle's position is defined by $s(t) = 3t^4 - 8t^2 + t + 1$, where position is measured in meters and time is measured in seconds.

   (a) Find an expression for the particle's velocity, $v(t)$, and acceleration, $a(t)$.

   (b) What is the particle's average velocity between $t = 0$ and $t = 2$?

   (c) When is the particle's velocity 1 m/s?

   (d) When is the particle's acceleration 20 m/s^2?

8.2 Related rates

1. An oil spill spreads out from a derrick in a circle, maintaining the circular shape as it spreads. If the radius of the slicks is expanding at a rate 3 km/hour, at what rate is the area of the slick changing when the radius is 10 km?

2. A perfectly spherical snowball begins to melt, maintaining its circular shape as it does so. If the volume of the snowball is decreasing at a rate of 2cm^3/minute, at what rate is the radius decreasing when the radius is 10 cm?

3. A plane flies over a radar station, and continues on its way. The plane is flying horizontally at an altitude of 3 kilometres, at a constant speed of 500 km/hour. Find the rate at which the distance between the plane and radar station is changing when the distance is 5 km.

4. A 5 metre long ladder leans up against a vertical wall. The bottom of the ladder begins to slide along the ground (away from the wall) at a speed of 2 m/s. How fast is the angle between the top of the ladder and the wall changing when the angle is 60°?

5. Imagine a right circular cylinder, which has a height growing at 2 cm/s and a radius shrinking at 1 cm/s. Is the volume increasing or decreasing when the radius and height are both 10 cm? By how much?

6. Snow is being blown into a conical pile at a rate of 10 m^3/minute. (Through careful maneuvering, the pile is always in the shape of a cone whose base radius is always half its height.) How fast is the snow pile increasing when the the pile is 5 metres high?

7. Two cars begin leave a restaurant at the same time. Car A travels due south at 50 km/hour, and car B travels due east at 40 km/hour. At what rate is the distance between the cars increasing after one hour?

8. A 2 metre tall man walks away from a lamppost at a speed of 1.5 m/s. If the lamppost is 4 metres tall, at what rate is the man's shadow changing in length when the man is 3 metres from the lamppost?

9. A rocket blast off from its launch pad at a speed of 1500 km/h. Just 20 kms away, a professor tracks the rocket with a telescope. Find the rate at which the angle between the telescope and the ground is changing 2 minutes after the launch.

10. The Batmobile (heading due south) is chasing the Jokermobile (heading due east). The Batmobile is heading towards a four-way intersection, and the Jokermobile is heading directly away from it. The Batmobile is moving at 150 km/h, while the Jokermobile is only moving at 130 km/h. Initially, the Batmobile is 30 km north of the intersection, and the Jokermobile is 40 km east of the intersection. At what rate is the distance between the two cars changing initially? At what rate is the distance changing after 5 minutes?

9 Extrema and how derivatives affect the shape of a graph

9.1 Extrema and the Mean Value Theorem

1. Find a point c satisfying the conclusion of the Mean Value Theorem for the function $f(x) = \sqrt{5-x}$ on the interval $[-4, 1]$.

2. Find the critical points for each of the following functions.

   (a) $f(x) = 2x^3 - 9x^2 + 12x$

   (b) $f(x) = \dfrac{4}{x^2 - 9}$

   (c) $f(x) = \dfrac{2(x+1)}{x^2}$

   (d) $f(x) = \dfrac{x^3}{9x + 18}$

   (e) $f(x) = x - \dfrac{x}{4-x}$

   (f) $f(x) = x^{\frac{2}{3}}(x-7)$

   (g) $f(x) = \arctan(x^2)$

9.2 The first derivative and the graph of a function

1. For each of the following functions, indicate the intervals where the function is increasing and decreasing and classify all local extrema.

(a) $f(x) = 2x^3 - 9x^2 + 12x$

(b) $f(x) = \dfrac{4}{x^2 - 9}$

(c) $f(x) = \dfrac{2(x+1)}{x^2}$

(d) $f(x) = \dfrac{x^3}{9x + 18}$

(e) $f(x) = x - \dfrac{x}{4 - x}$

(f) $f(x) = x^{\frac{2}{3}}(x - 7)$

(g) $f(x) = \arctan(x^2)$

9.3 The second derivative and the graph of a function

1. For each of the following, find the inflection points and indicate the intervals where the function is increasing and decreasing.

   (a) $f(x) = 2x^3 - 9x^2 + 12x$

   (b) $f(x) = \dfrac{4}{x^2 - 9}$

   (c) $f(x) = \dfrac{2(x+1)}{x^2}$

   (d) $f(x) = \dfrac{x^3}{9x + 18}$

   (e) $f(x) = x - \dfrac{x}{4 - x}$

   (f) $f(x) = x^{\frac{2}{3}}(x - 7)$

   (g) $f(x) = \arctan(x^2)$

2. Use the Second Derivative Test to find the local maxima and minima for the following functions.

   (a) $f(x) = 2x^3 - 9x^2 + 12x$

   (b) $f(x) = 3x^4 + 4x^3 + 1$

10 Limits at infinity and curve sketching

10.1 Limits at infinity and horizontal asymptotes

1. Find the following limits, if they exist. If they do not, but approach $\pm\infty$, indicate this.

   (a) $\displaystyle\lim_{x\to\infty} \frac{7 + 3x^2}{1 + 10x^2 + 4x^3}$

(b) $\lim_{x\to-\infty} \dfrac{1+3x^6}{x^4+1}$

(c) $\lim_{x\to\infty} \dfrac{7-3x+4x^2}{9-2x-3x^2}$

(d) $\lim_{x\to-\infty} \dfrac{5+3x+2x^2}{3-x}$

(e) $\lim_{x\to\infty} \dfrac{7-2x-x^2}{x-5}$

(f) $\lim_{x\to\infty} \dfrac{3x^2-5}{8x^4+1}$

(g) $\lim_{x\to-\infty} \dfrac{3x^2-5}{8x^4+1}$

(h) $\lim_{x\to\infty} \dfrac{3x^3-5}{8x^6+1}$

(i) $\lim_{x\to-\infty} \dfrac{3x^3-5}{8x^6+1}$

2. For each of the following functions, indicate the horizontal asymptotes, if they exist.

   (a) $f(x) = 2x^3 - 9x^2 + 12x$

   (b) $f(x) = \dfrac{4}{x^2-9}$

   (c) $f(x) = \dfrac{2(x+1)}{x^2}$

   (d) $f(x) = \dfrac{x^3}{9x+18}$

   (e) $f(x) = x - \dfrac{x}{4-x}$

   (f) $f(x) = x^{\frac{2}{3}}(x-7)$

   (g) $f(x) = \arctan(x^2)$

10.2 Vertical asymptotes and symmetry

1. For each of the following functions, indicate the vertical asymptotes, if they exist. Clearly indicate the behaviour of the function on either side of the asymptote.

   (a) $f(x) = 2x^3 - 9x^2 + 12x$

   (b) $f(x) = \dfrac{4}{x^2-9}$

   (c) $f(x) = \dfrac{2(x+1)}{x^2}$

   (d) $f(x) = \dfrac{x^3}{9x+18}$

(e) $f(x) = x - \dfrac{x}{4-x}$

(f) $f(x) = x^{\frac{2}{3}}(x-7)$

(g) $f(x) = \arctan(x^2)$

Not sure about symmetry. Don't see anything in the notes about it?

10.3 Summary of curve sketching

1. Sketch the following functions.

   (a) $f(x) = 2x^3 - 9x^2 + 12x$

   (b) $f(x) = \dfrac{4}{x^2-9}$

   (c) $f(x) = \dfrac{2(x+1)}{x^2}$

   (d) $f(x) = \dfrac{x^3}{9x+18}$

   (e) $f(x) = x - \dfrac{x}{4-x}$

   (f) $f(x) = x^{\frac{2}{3}}(x-7)$

   (g) $f(x) = \arctan(x^2)$

11 Optimization problems

11.1 The closed interval method for extrema

1. Find the absolute extrema of the following functions on the given interval.

   (a) $f(x) = x\sqrt{4-x^2}$, on $[-1, 2]$

   (b) $f(x) = x^2\sqrt{7-x}$, on $[-2, 7]$

   (c) $f(x) = \sin(x) - \cos(x)$, on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

   (d) $f(x) = xe^{-x}$, on $[0, 2]$

2. A piece of wire of length 100 cm is cut into two pieces, each of which is bent into a square. Into what lengths should the wire be cut to minimize the sum of the areas of the two squares?

3. The legs of a right triangle have lengths a and b such that the sum of their lengths is 10. What is the maximum area of such a triangle?

4. Tara is trying to get to her cabin. Her cabin is 4 km from the closest point on the main road. That point is 9 km along the perfectly straight main road from the highway. Tara can drive 50 km/h on the main road, but will eventually park her car to walk 5 km/h overland. Where on the main road should Tara park her car to minimize the amount of time it will take her to get to her cabin, assuming that she walks in a straight line from the point she leaves the main road?

11.2 Optimization problems

1. Find the point P on the parabola $y = x^2$ closest to the point $(3, 0)$.

2. A gardener wishes to enclose a rectangular garden on one side by a brick wall costing \$3/m and on the other three sides by a metal fence costing \$1/m. If the area of the garden is 18 m^2, find the minimum cost of building the enclosure.

3. A farmer wishes to enclose a rectangular field which will be divided in half by another fence that runs parallel to one side. If the farmer only has 600 m of fencing material, what is the maximum area she can fence?

4. A poster is to have a printed area of 400 square centimetres with a blank 8 cm border top and bottom as well as a blank 5 cm border on both sides. Find the dimensions of the poster that will have the smallest perimeter.

5. A cylindrical can, closed on top and bottom, is to be constructed using 600π square centimeters of material. What is the maximum volume of such a can?

6. A closed box with a square base is to be constructed with a volume of 20 cubic metres. If the material for the base costs \$3 per square metre, the material for the top costs \$2 per square metre, and the material for the sides costs \$1 per square metre, what dimensions will minimize the cost of building the box?

7. Find the dimensions of the largest rectangle that can be inscribed with one side on the x-axis and the other two corners on the parabola $y = 16 - x^2$.

11.3 Examples of optimization problems

12 Indeterminate forms and L'Hôpital's Rule

12.1 Indeterminate forms and L'Hôpital's Rule

1. Use L'Hôpital's Rule to find the following limits.

   (a) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)}$

   (c) $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{x-1}$

   (d) $\displaystyle\lim_{x\to\infty} \frac{(\ln(x))^3}{x^3+1}$

   (e) $\displaystyle\lim_{x\to 1^+} \frac{x}{x-1} - \frac{1}{\ln(x)}$

12.2 L'Hôpital's Rule for products and exponents

1. Use L'Hôpital's Rule to find the following limits.

   (a) $\lim_{x \to -\infty} xe^x$

   (b) $\lim_{x \to 1^+} (x-1) \tan\left(\frac{\pi x}{2}\right)$

   (c) $\lim_{x \to 0^+} (\cos(x))^{\frac{1}{x^2}}$

   (d) $\lim_{x \to \infty} xe^{\frac{1}{x}} - x$

   (e) $\lim_{x \to 0^+} \sin(x)^{\tan(x)}$

   (f) $\lim_{x \to 0^+} x^{x^2}$

   (g) $\lim_{x \to \infty} \left(1 + 4e^{2x}\right)^{\frac{3}{x}}$

   (h) $\lim_{x \to \infty} \left(1 + \frac{5}{x}\right)^{3x}$